Project systems theory – Solutions

Resit exam 2017–2018, Thursday 12 April 2018, 14:00 – 17:00

Problem 1

(4 + 8 = 12 points)

A simple model for the metabolism of alcohol in the body is given by

$$\begin{aligned} \dot{c}_b(t) &= q_b\big(c_l(t) - c_b(t)\big) \\ \dot{c}_l(t) &= q_l\big(c_b(t) - c_l(t)\big) - \phi\big(c_l(t)\big) + u(t) \end{aligned} \tag{1}$$

where $c_b(t)$ and $c_l(t)$ are the concentrations of alcohol in the body and liver, respectively. The intake of alcohol is given by the input $u(t)$ and the function

$$\phi(c_l) = q_{\max}\frac{c_l}{c_0 + c_l} \tag{2}$$

gives the rate at which the liver reduces the alcohol concentration. The constants $q_b, q_l, q_{\max}$, and c_0 are all positive.

a) The equilibrium point $(\bar{c}_b, \bar{c}_l)$ for $u(t) = \bar{u}$ is obtained by solving (1) for $\dot{c}_b = 0$ and $\dot{c}_l = 0$. It then immediately follows from the first equation that $\bar{c}_l = \bar{c}_b$, after which the substitution of this result in the second equation of (1) yields

$$q_{\max}\frac{\bar{c}_l}{c_0 + \bar{c}_l} = \bar{u}, \tag{3}$$

where the definition of ϕ in (2) is used. Solving (3) for $\bar{c}_l$ gives the final result

$$\bar{c}_b = \bar{c}_l = \frac{c_0\bar{u}}{q_{\max} - \bar{u}}. \tag{4}$$

Note that the assumption $\bar{u} < q_{\max}$ implies that the equilibrium is well-defined.

b) Before finding the linearized dynamics, let x denote that state of (1) as

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} c_b \\ c_l \end{bmatrix} \tag{5}$$

and define the function f to be the corresponding vector field, i.e.,

$$f(x, u) = \begin{bmatrix} q_b(x_2 - x_1) \\ q_l(x_1 - x_2) - \phi(x_2) + u \end{bmatrix}. \tag{6}$$

After defining the perturbations

$$\tilde{x} = x - \bar{x}, \qquad \tilde{u} = u - \bar{u}, \tag{7}$$

with $\bar{x} = [\,\bar{c}_b \;\; \bar{c}_l\,]^{\mathrm{T}}$, the linearized dynamics is given as

$$\dot{\tilde{x}}(t) = \frac{\partial f}{\partial x}(\bar{x}, \bar{u})\tilde{x}(t) + \frac{\partial f}{\partial u}(\bar{x}, \bar{u})\tilde{u}(t). \tag{8}$$

Then, it follows from (6) that

$$\frac{\partial f}{\partial x}(x, u) = \begin{bmatrix} -q_b & q_b \\ q_l & -q_l - \frac{\mathrm{d}\phi}{\mathrm{d}x_2}(x_2) \end{bmatrix}, \tag{9}$$

with

$$\frac{\mathrm{d}\phi}{\mathrm{d}x_2}(x_2) = q_{\max}\frac{c_0}{(c_0 + x_2)^2}, \tag{10}$$

such that

$$\frac{\partial f}{\partial x}(\bar{x}, \bar{u}) = \begin{bmatrix} -q_b & q_b \\ q_l & -q_l - q_{\max}\frac{c_0}{(c_0+\bar{x}_2)^2} \end{bmatrix}. \tag{11}$$

Moreover, it is immediate that

$$\frac{\partial f}{\partial u}(\bar{x}, \bar{u}) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{12}$$

Problem 2 (16 points)

Consider the linear system

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -8 & -4a & -b & -a \end{bmatrix} x(t), \tag{13}$$

where $a, b \in \mathbb{R}$.

To determine stability of (13), denote

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -8 & -4a & -b & -a \end{bmatrix}, \tag{14}$$

and note that A is a so-called companion matrix. Consequently, its characteristic polynomial is obtained immediately as

$$\Delta_A(\lambda) = \lambda^4 + a\lambda^3 + b\lambda^2 + 4a\lambda + 8. \tag{15}$$

Now, to determine the values of a, b for which the system (13) is stable (or, equivalently, for which the polynomial (15) is stable), we will use the Routh-Hurwitz test.

However, before setting up the Routh-Hurwitz table, it is recalled that a necessary condition for a polynomial to be stable is that all its coefficient have the same sign (and are nonzero). This implies in particular that

$$a > 0, \qquad b > 0. \tag{16}$$

To proceed, consider the following Routh-Hurwitz table:

$$\begin{array}{r|ccccc|l}
 & \lambda^4 & \lambda^3 & \lambda^2 & \lambda^1 & \lambda^0 & \\
\hline
a\times & 1 & a & b & 4a & 8 & \\
1\times & a & & 4a & & & \\
\hline
 & & a^2 & a(b-4) & 4a^2 & 8a & \text{(result of Step 1)} \\
\hline
(b-4)\times & & a & (b-4) & 4a & 8 & \text{(result of Step 2)} \\
a\times & & (b-4) & & 8 & & \\
\hline
4a(b-6)\times & & & (b-4)^2 & 4a(b-6) & 8(b-4) & \text{(result of Step 3)} \\
(b-4)^2\times & & & 4a(b-6) & & & \\
\hline
\end{array} \tag{17}$$

Recall that the Routh-Hurwitz criterion states that the polynomial Δ_A in (15) if and only if its two leading coefficients have the same sign and that the polynomial obtained in Step 1 is stable. Given (16), the coefficients 1 and a satisfy the first statement. Then, using a similar reasoning as before, it is necessary that

$$b - 4 > 0 \quad \Leftrightarrow \quad b > 4 \tag{18}$$

in order for the polynomial that results from Step 1 to be stable. Now we thus have $a > 0$ and $b > 4$.

Note that, as $a > 0$ is a necessary condition for stability, the polynomial that results from Step 1 can be dived by a to obtain the result of Step 2.

Next, applying the Routh-Hurwitz criterion to the result of Step 2 leads to the result of Step 3. Clearly, this gives

$$b > 6 \tag{19}$$

as a necessary condition for stability, such that we obtain $a > 0$ and $b > 6$.

Repeating this procedure gives the result of Step 4 (not listed in the table) as the polynomial

$$p(\lambda) = \big(4a(b-6)\big)\big(4a(b-6)\lambda + 8(b-4)\big), \tag{20}$$

which only root is computed as

$$\lambda = -\frac{8(b-4)}{4a(b-6)}. \tag{21}$$

Recall that necessary conditions for stability are given by $a > 0$ and $b > 6$. However, under these conditions, it is readily verified that $\lambda < 0$, i.e., the polynomial p is stable. Consequently, the original polynomial (15) is stable if and only if

$$a > 0, \qquad b > 6. \tag{22}$$

Problem 3

(4 + 12 + 6 = 22 points)

Consider the system

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{23}$$

with state $x(t) \in \mathbb{R}^2$, input $u(t) \in \mathbb{R}$, and where

$$A = \begin{bmatrix} -7 & -4 \\ 4 & 3 \end{bmatrix}, \qquad B = \begin{bmatrix} -3 \\ 2 \end{bmatrix}. \tag{24}$$

a) Controllability of the system (23)-(24) can be verified by computing

$$\begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} -3 & 13 \\ 2 & -6 \end{bmatrix}, \tag{25}$$

which is easily verified to have rank 2. Consequently, the system is controllable.

b) Since, by the result of a), the system (23)-(24) is controllable, there exists a nonsingular matrix T such that

$$T^{-1}AT = \begin{bmatrix} 0 & 1 \\ \alpha_1 & \alpha_2 \end{bmatrix}, \qquad T^{-1}B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

for some real α_1 and α_2. This is in fact the controllable canonical form and the numbers α_1 and α_2 equal (but with negative sign) the coefficients of the characteristic polynomial of A in (23). Therefore, we compute

$$\begin{aligned} \Delta_A(\lambda) = \det(\lambda I - A) &= \begin{vmatrix} \lambda + 7 & 4 \\ -4 & \lambda - 3 \end{vmatrix} && (26) \\ &= (\lambda + 7)(\lambda - 3) + 16 && (27) \\ &= \lambda^2 + 4\lambda - 5 && (28) \\ &= \lambda^2 + a_1\lambda + a_2, && (29) \end{aligned}$$

such that

$$a_1 = 4, \qquad a_2 = -5, \tag{30}$$

and, in particular,

$$\alpha_1 = -a_2 = 5, \qquad \alpha_2 = -a_1 = -4. \tag{31}$$

The corresponding transformation T can be constructed by computing the vector q_2 as

$$q_2 = B = \begin{bmatrix} -3 \\ 2 \end{bmatrix} \tag{32}$$

as well as the vector q_1 given by

$$q_1 = AB + a_1 B = \begin{bmatrix} 13 \\ -6 \end{bmatrix} + 4 \begin{bmatrix} -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}. \tag{33}$$

Here, note that the matrix-vector product AB was already computed in (25). Then, the matrix T is obtained as

$$T = \begin{bmatrix} q_1 & q_2 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 2 & 2 \end{bmatrix}, \tag{34}$$

whereas its inverse can be computed to be

$$T^{-1} = \frac{1}{8}\begin{bmatrix} 2 & 3 \\ -2 & 1 \end{bmatrix}. \tag{35}$$

Then, by direct computation, it is verified that

$$T^{-1}AT = \frac{1}{8}\begin{bmatrix} 2 & 3 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} -7 & -4 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 1 & -3 \\ 2 & 2 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 2 & 3 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} -15 & 13 \\ 10 & -6 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 5 & -4 \end{bmatrix}, \tag{36}$$

$$T^{-1}B = \frac{1}{8}\begin{bmatrix} 2 & 3 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{37}$$

which is indeed the desired form.

c) To place the eigenvalues of $A + BF$ at -1 and -2, define the polynomial p that has these eigenvalues as its roots. This polynomial is given as

$$p(\lambda) = (\lambda + 1)(\lambda + 2) = \lambda^2 + 3\lambda + 2. \tag{38}$$

After defining $\bar{A} = T^{-1}AT$ and $\bar{B} = T^{-1}B$ (see the results (36) and (37), respectively) and introducing the matrix

$$\bar{F} = \begin{bmatrix} \bar{F}_1 & \bar{F}_2 \end{bmatrix}, \tag{39}$$

we obtain

$$\bar{A} + \bar{B}\bar{F} = \begin{bmatrix} 0 & 1 \\ 5 + \bar{F}_1 & -4 + \bar{F}_2 \end{bmatrix}. \tag{40}$$

The matrix $\bar{A} + \bar{B}\bar{F}$ has the characteristic polynomial

$$\Delta_{\bar{A}+\bar{B}\bar{F}}(\lambda) = \lambda^2 + (4 - \bar{F}_2)\lambda - (5 + \bar{F}_1). \tag{41}$$

Matching coefficients of (38) and (41) leads to

$$\bar{F}_1 = -7, \qquad \bar{F}_2 = 1. \tag{42}$$

After observing that

$$T\big(\bar{A} + \bar{B}\bar{F}\big)T^{-1} = A + B\bar{F}T^{-1}, \tag{43}$$

it is clear that the desired feedback matrix F is given as

$$F = \bar{F}T^{-1} = \tfrac{1}{8}\begin{bmatrix} -7 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ -2 & 1 \end{bmatrix} = \tfrac{1}{8}\begin{bmatrix} -16 & -20 \end{bmatrix} = \begin{bmatrix} -2 & -\tfrac{5}{2} \end{bmatrix}. \tag{44}$$

Problem 4 (3 + 3 + 3 + 3 + 4 + 6 = 22 points)

Consider the system

$$\dot{x}(t) = \begin{bmatrix} -2 & -1 & 0 \\ 1 & -2 & 0 \\ 6 & -4 & 3 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(t), \qquad y(t) = [\,1 \ -1 \ 1\,]x(t), \tag{45}$$

and denote for future reference

$$A = \begin{bmatrix} -2 & -1 & 0 \\ 1 & -2 & 0 \\ 6 & -4 & 3 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad C = [\,1 \ -1 \ 1\,]. \tag{46}$$

a) Stability of (45) is determined by the spectrum $\sigma(A)$ of A in (46), which, due to its block lower triangular structure, is given as

$$\sigma(A) = \sigma\left(\begin{bmatrix} -2 & -1 \\ 1 & -2 \end{bmatrix}\right) \cup \{3\}. \tag{47}$$

As $3 \in \sigma(A)$, it is clear that (45) is not asymptotically stable.

We will also compute the full spectrum of A. The eigenvalues of the upper-left block are given as the roots of

$$\begin{vmatrix} \lambda + 2 & 1 \\ -1 & \lambda + 2 \end{vmatrix} = (\lambda + 2)^2 + 1 = 0, \tag{48}$$

which implies that $\lambda + 2 = \pm i$. Consequently, its roots read $-2 + i$ and $-2 - i$, such that

$$\sigma(A) = \{-2 + i, -2 + i, 3\}. \tag{49}$$

b) A direct computation of the controllability matrix yields

$$\begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & -4 \\ 0 & 6 & 2 \end{bmatrix}, \tag{50}$$

which is observed to have rank three. Thus, the system (45) is controllable.

c) The system is stabilizable as this is implied by controllability (see problem b)).

d) To determine whether the system is observable, compute

$$\begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 3 & -3 & 3 \\ 9 & -9 & 9 \end{bmatrix}. \tag{51}$$

As all rows are scaled versions of the first row, it is immediate that

$$\operatorname{rank} \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = 1 < 3, \tag{52}$$

and the system is not observable.

e) The unobservable subspace $\mathcal{N}$ is given by

$$\mathcal{N} = \ker \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix}, \tag{53}$$

Using (51), it follows that

$$\mathcal{N} = \text{span}\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}\} \tag{54}$$

is a basis for $\mathcal{N}$

For the final question in Problem 4, consider the system

$$\dot{x}(t) = \begin{bmatrix}2 & 2-a & 1-a\\0 & a & 1\\0 & 0 & a\end{bmatrix} x(t), \qquad y(t) = \begin{bmatrix}1 & 1 & 1\end{bmatrix} x(t) \tag{55}$$

with $a \in \mathbb{R}$. Denote

$$A = \begin{bmatrix}2 & 2-a & 1-a\\0 & a & 1\\0 & 0 & a\end{bmatrix}, \qquad C = \begin{bmatrix}1 & 1 & 1\end{bmatrix}. \tag{56}$$

f) By the Hautus test, (55) is detectable if and only if the following implication holds

$$\lambda \in \sigma(A),\ \text{Re}(\lambda) \geq 0 \qquad \Longrightarrow \qquad \text{rank} \begin{bmatrix}\lambda I - A\\C\end{bmatrix} = n \tag{57}$$

where n is the dimension of the state space of (55), i.e., $n = 3$.

To evaluate the condition (57), it is first remarked that the upper triangular structure of A immediately reveals its spectrum as

$$\sigma(A) = \{2, a\}. \tag{58}$$

Now, the evaluation of (57) for $\lambda = 2$ (as $\text{Re}(2) \geq 0$) gives

$$\begin{bmatrix}\lambda I - A\\C\end{bmatrix} = \begin{bmatrix}\lambda-2 & a-2 & a-1\\0 & \lambda-a & -1\\0 & 0 & \lambda-a\\1 & 1 & 1\end{bmatrix} = \begin{bmatrix}0 & a-2 & a-1\\0 & 2-a & -1\\0 & 0 & 2-a\\1 & 1 & 1\end{bmatrix}, \tag{59}$$

which has full rank if and only if $a \neq 2$.

If $a < 0$, it is clear that the condition (57) only needs to be verified for $\lambda = 2$, in which case the result (59) shows that (55) is detectable (as $a < 0$ implies that $a \neq 2$).

It remains to be verified if there exist $a \geq 0$ for which (55) is detectable. To this end, consider (57) for $\lambda = a$ to obtain

$$\begin{bmatrix}\lambda I - A\\C\end{bmatrix} = \begin{bmatrix}\lambda-2 & a-2 & a-1\\0 & \lambda-a & -1\\0 & 0 & \lambda-a\\1 & 1 & 1\end{bmatrix} = \begin{bmatrix}a-2 & a-2 & a-1\\0 & 0 & -1\\0 & 0 & 0\\1 & 1 & 1\end{bmatrix}. \tag{60}$$

As the sum of the first two rows is a multiple of the last row, it is clear that the matrix on the right-hand side has rank two for each a. Thus, the eigenvalue $\lambda = a$ is never observable and we conclude that (55) is detectable if and only if $a < 0$.

Problem 5 (6 + 12 = 18 points)

Consider the discrete-time system

$$x_{k+1} = Ax_k + Bu_k, \tag{61}$$

with state $x_k \in \mathbb{R}^n$ and input $u_k \in \mathbb{R}^m$.

a) We claim that the solution of (61) for initial condition $x_0 \in \mathbb{R}^n$ and input sequence $\{u_0, u_1, \dots\}$ is given by

$$x_k = A^k x_0 + \sum_{i=0}^{k-1} A^{k-i-1} B u_i. \tag{62}$$

To show this, note that the evaluation of (62) for $k = 1$ immediately yields (61). It remains to be verified that (62) satisfies the discrete dynamics (61) for $k > 1$. Substitution of the expression (62) for x_k in the dynamics (61) gives

$$x_{k+1} = A\left(A^k x_0 + \sum_{i=0}^{k-1} A^{k-i-1} B u_i\right) + Bu_k, \tag{63}$$

$$= A^{k+1} x_0 + \sum_{i=0}^{k-1} A^{(k+1)-i-1} B u_i + Bu_k, \tag{64}$$

$$= A^{k+1} x_0 + \sum_{i=0}^{(k+1)-1} A^{(k+1)-i-1} B u_i, \tag{65}$$

where it is remarked that, in the sum in (65), the term for $i = (k+1) - 1 = k$ reads $A^0 B u_k = B u_k$. It is clear that the result (65) is again of the form (62) (but for index $k+1$ instead of k), showing that the general form (62) satisfies the dynamics.

A discrete-time system (61) is said to be *controllable* if, for every initial condition $x_0 \in \mathbb{R}^n$ and every final state $\bar{x} \in \mathbb{R}^n$, there exists an integer $K > 0$ and an input sequence $\{u_0, u_1, \dots, u_{K-1}\}$ such that $x_K = \bar{x}$, with x_K the solution at step K as in (62).

b) To prove that (61) is controllable if and only if

$$\operatorname{rank}\begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix} = n, \tag{66}$$

note that (62) can be written as

$$x_k - A^k x_0 = \begin{bmatrix} B & AB & \cdots & A^{k-2}B & A^{k-1}B \end{bmatrix} \begin{bmatrix} u_{k-1} \\ u_{k-2} \\ \vdots \\ u_1 \\ u_0 \end{bmatrix}. \tag{67}$$

In the remainder of this proof, sufficiency and necessity are proven separately.

if) Let (66) hold and consider (67) for $k = n$. As (66) implies that

$$\operatorname{im}\begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix} = \mathbb{R}^n, \tag{68}$$

it follows that, for every $x_0 \in \mathbb{R}^n$, $\bar{x} \in \mathbb{R}^n$, there exists an input sequence $\{u_0, u_1, \dots, u_{n-1}\}$ such that $x_n = \bar{x}$, i.e., the discrete-time system is controllable.

only if) Let (61) be controllable. Then, for any $x_0 \in \mathbb{R}^n$ and $\bar{x} \in \mathbb{R}^n$, there exists an integer K and an input sequence $\{u_0, u_1, \ldots, u_{K-1}\}$ such that (67) holds for $k = K$. Stated differently,

$$\bar{x} - A^K x_0 \in \operatorname{im} \left[\, B \;\; AB \;\; A^2B \;\; \cdots \;\; A^{K-1}B \, \right]. \tag{69}$$

We claim that this implies that

$$\bar{x} - A^K x_0 \in \operatorname{im} \left[\, B \;\; AB \;\; A^2B \;\; \cdots \;\; A^{n-1}B \, \right]. \tag{70}$$

Namely, for $K \leq n$, it is immediate

$$\operatorname{im} \left[\, B \;\; AB \;\; A^2B \;\; \cdots \;\; A^{K-1}B \, \right] \subset \operatorname{im} \left[\, B \;\; AB \;\; A^2B \;\; \cdots \;\; A^{n-1}B \, \right], \tag{71}$$

whereas the case $K > n$ follows from the theorem of Cayley-Hamilton. In this case,

$$\operatorname{im} \left[\, B \;\; AB \;\; A^2B \;\; \cdots \;\; A^{K-1}B \, \right] = \operatorname{im} \left[\, B \;\; AB \;\; A^2B \;\; \cdots \;\; A^{n-1}B \, \right]. \tag{72}$$

Thus, we have (70). As $\bar{x}_0$ and $\bar{x}$ are arbitrary, it follows that

$$\operatorname{im} \left[\, B \;\; AB \;\; A^2B \;\; \cdots \;\; A^{n-1}B \, \right] = \mathbb{R}^n, \tag{73}$$

(take, e.g., $x_0 = 0$ and arbitrary $\bar{x}$) which is equivalent to the rank condition (66).